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Dihedral singularities and gravitational instantons

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We produce a family of gravitational instantons which arise as deformations of dihedral singularities.

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1. Introduction

A gravitational instanton is a four-manifold with a nonsingular Riemannian metric satisfying Einstein's equations. A special class of gravitational instantons is that of hyperkähler four-manifolds. The hyperkähler condition means that there are three covariant constant complex structures I, J and K (with respect to which the metric is hermitian), which satisfy the quaternionic multiplication relations. This implies the existence of a whole two-sphere's worth of Kähler structures on the manifold. Hyperkähler manifolds are Ricci-flat, so in particular are Einstein.

Kronheimer [K1, K2] has carried out a classification of hyperkähler four-manifolds which satisfy the Asymptotically Locally Euclidean (ALE) condition. This means that the metric looks asymptotically like the quotient by a finite group of Euclidean $\mathbb{R}^4$. These ALE spaces turn out to be intimately related to the Kleinian singularities of algebraic geometry, which we now explain.

Let Γ be a finite subgroup of SU(2). Such groups have been classified and Γ must be of one of the following types:

(i) the cyclic group of order k,
(ii) the binary dihedral group of order $4k$,
(iii) the binary tetrahedral group of order 24,
(iv) the binary octahedral group of order 48,
(v) the binary icosahedral group of order 120.

If we view $\mathbb{C}^2$ as the standard two-dimensional representation of SU(2), it follows that Γ acts on $\mathbb{C}^2$ fixing the origin. The quotient $\mathbb{C}^2/\Gamma$ is then a singular complex surface, which can be identified with one of the following hypersurfaces (the *Kleinian singularities*) in $\mathbb{C}^3$:

(i) $xy=z^k$,

(ii) $x^2-zy^2=z^{k+1}$,

(iii) $x^2+y^3+z^4=0$,

(iv) $x^2+y^3+yz^3=0$,

(v) $x^2+y^3+z^5=0$.

These singularities [and the finite subgroups of SU(2)] correspond to the simply laced Dynkin diagrams A_k, D_k, E_6, E_7, E_8.

Kronheimer [K1, K2] showed that each of these singular spaces admitted deformations carrying ALE hyperkähler metrics, and that all ALE hyperkähler four-manifolds arose in this way. The manifolds arising from the cyclic group singularities (i) are in fact the multi-instanton spaces of Gibbons and Hawking [GH, H1].

Now, one can also consider hyperkähler spaces which are not ALE. One well known example is the multi-Taub–NUT series of metrics which are Asymptotically Locally Flat (ALF). This condition means that the metric approaches

$$ds^2=dr^2+r^2(\rho_1^2+\rho_2^2)+c^2\rho_3^2$$

at least as fast as $1/r$; here c is a constant and ρ_1, ρ_2, ρ_3 are left-invariant one-forms on the quotient of S^3 by a finite group Γ.

Now the multi-Taub–NUT metrics also live on deformations of the cyclic singularities $xy=z^k$, so can be viewed as a non-ALE counterpart to the multi-instanton series.

In this paper we shall present a family of hyperkähler four-manifolds which live on deformations of the dihedral singularities. We shall obtain them by a modification of Kronheimer's construction, analogous to the way in which the multi-Taub–NUT sequence is obtained as a modification of the multi-instanton construction. This construction will involve the method of hyperkähler quotients, which we describe in the next section.

2. Hyperkähler quotients

If M is a hyperkähler manifold with metric h and complex structures I, J and K, then it carries three Kähler forms ω_1, ω_2, ω_3 defined by

$$\omega_1(X, Y)=h(IX, Y) ,$$

etc. Suppose now that a group G acts on M preserving h. Assume moreover that the action of G preserves I, J, K. We express the latter condition by saying that the action is triholomorphic. If G is compact we can define three G-equivariant moment maps μ_1, μ_2, μ_3 from M to the dual of the Lie algebra $\mathfrak{g}$ of G by

$$\langle \mu_i(m), \zeta \rangle = \Phi_i^{\zeta}(m) \quad (m \in M, \zeta \in \mathfrak{g}) ,$$

where $d\Phi_i^{\zeta}$ is the contraction of ω_i with ζ, and we identify an element of $\mathfrak{g}$ with the Killing field it generates on M. We can combine these moment maps into a single map μ taking values in $\mathbb{R}^3 \otimes \mathfrak{g}^*$.

Theorem 2.1 [HKLR]. *If $\lambda_1, \lambda_2, \lambda_3$ lie in the centre $\mathscr{Z}$ of $\mathfrak{g}^*$, and if G acts freely on $\mu^{-1}(\lambda_1, \lambda_2, \lambda_3)$ then $\mu^{-1}(\lambda_1, \lambda_2, \lambda_3)/G$ is hyperkähler.* □

We refer to this manifold as the *hyperkähler quotient* of M by G. The space $\mu^{-1}(\lambda_1, \lambda_2, \lambda_3)$ is called the *level set*.

The following result is often useful in identifying hyperkähler quotients as complex manifolds.

Theorem 2.2 [HKLR]. *The hyperkähler quotient $\mu^{-1}(0, \lambda_2, \lambda_3)/G$ is isomorphic as a complex manifold to the quotient by $G_{\mathbb{C}}$ (the complexification of G) of the open set of stable points in $(\mu_2 + i\mu_3)^{-1}(\lambda_2 + i\lambda_3)$.* □

This is a statement of the general principle of equivalence of Kähler and algebro-geometric quotients, applied to the complex manifold $(\mu_2 + i\mu_3)^{-1}(\lambda_2 + i\lambda_3)$.

The multi-instanton spaces may be obtained using a hyperkähler quotient construction as follows. We denote the space of quaternions by H. Let $M = H^k$ and let $G = \mathrm{U}(1)^k$ act on M isometrically by

$$x_1 \mapsto x_1 e^{i(\theta_1 - \theta_2)} , \quad \ldots, \quad x_k \mapsto x_k e^{i(\theta_k - \theta_1)} , \tag{1}$$

where $x_i \in H$. Note that this is really an action of $\mathrm{U}(1)^{k-1}$, as the circle subgroup of G for which all θ_i are equal acts trivially.

We take the quaternionic structure on H to be defined by multiplication on the left by unit imaginary quaternions, so the above action is triholomorphic. We can therefore apply the quotient construction of theorem 2.1.

If we take the level set to be $\mu^{-1}(0, 0, 0)$ the hyperkähler quotient is the singular space $xy = z^k$. Varying the level set gives hyperkähler deformations of this singularity, which are the multi-instanton spaces of Gibbons and Hawking.

We can modify the above construction by replacing one of the Euclidean spaces H by the flat (but non-Euclidean) hyperkähler space $\mathbb{R}^3 \times S^1$. The triholomorphic circle action on $\mathbb{R}^3 \times S^1$ is rotation in the S^1 factor. The quotient construction now gives us the multi-Taub–NUT spaces, which are deformations of $xy = z^k$ carrying non-ALE metrics.

3. A quotient construction

In the spirit of the preceding remarks we shall now modify the construction of the dihedral ALE spaces due to Kronheimer. We first review some material about actions of unitary groups on quaternionic spaces.

Recall first that we have a triholomorphic isometric action of U(2) on H^2 given by

$$(x_1, x_2) \mapsto (x_1, x_2) A^{\mathrm{T}} .$$

If we choose a complex structure on H^2 then this action becomes

$$(R, S) \mapsto (AR, SA^{-1}) ,$$

where R, S are elements of $\mathrm{Hom}(\mathbb{C}, \mathbb{C}^2)$ and $\mathrm{Hom}(\mathbb{C}^2, \mathbb{C})$, respectively. The complex moment map is

$$\mu_2 + \mathrm{i}\mu_3 : (R, S) \mapsto RS . \tag{2}$$

Note that this has the correct equivariance properties.

Next, observe that there is a triholomorphic isometric action of U(2)×U(2) on H^4 defined by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4) \begin{pmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_1 & 0 & a_3 \\ a_2 & 0 & a_4 & 0 \\ 0 & a_2 & 0 & a_4 \end{pmatrix} \begin{pmatrix} \bar{a}_5 & \bar{a}_7 & 0 & 0 \\ \bar{a}_6 & \bar{a}_8 & 0 & 0 \\ 0 & 0 & \bar{a}_5 & \bar{a}_7 \\ 0 & 0 & \bar{a}_6 & \bar{a}_8 \end{pmatrix} ,$$

where

$$A_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a_5 & a_6 \\ a_7 & a_8 \end{pmatrix}$$

are elements of U(2). We denote this action by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4)(A_1, A_2) .$$

Choosing a complex structure on H^4 the U(2)×U(2) action becomes

$$(Z, W) \mapsto (A_1 Z A_2^{-1}, A_2 W A_1^{-1}) ,$$

where $Z, W \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2)$. Note that the circle of scalars in U(2)×U(2) acts trivially. The complex moment map for this action is

$$\mu_2 + \mathrm{i}\mu_3 : (Z, W) \mapsto (ZW, -WZ) , \tag{3}$$

taking values in $\mathfrak{gl}(2, \mathbb{C}) \times \mathfrak{gl}(2, \mathbb{C})$.

In his construction of the D_k dihedral ALE gravitational instantons, Kronheimer [K1] takes M to be $H^2 \times H^2 \times (H^4)^{k-4} \times H^2 \times H^2$ and G to be $\mathrm{U}(1)^4 \times \mathrm{U}(2)^{k-3}$. The action of G on M is

$$(x_1, x_2) \mapsto (x_1, x_2) A_1^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}\theta_1} ,$$

$$(x_3, x_4) \mapsto (x_3, x_4) A_1^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}\theta_2} ,$$

$$(x_5, x_6, x_7, x_8) \mapsto (x_5, x_6, x_7, x_8)(A_1, A_2) ,$$

$$(x_9, x_{10}, x_{11}, x_{12}) \mapsto (x_9, x_{10}, x_{11}, x_{12})(A_2, A_3) ,$$

$$\dots$$

$$(x_{4k-15}, x_{4k-14}, x_{4k-13}, x_{4k-12}) \mapsto (x_{4k-15}, x_{4k-14}, x_{4k-13}, x_{4k-12})(A_{k-4}, A_{k-3}) ,$$

$$(x_{4k-11}, x_{4k-10}) \mapsto (x_{4k-11}, x_{4k-10}) A_{k-3}^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}\theta_3} ,$$

$$(x_{4k-9}, x_{4k-8}) \mapsto (x_{4k-9}, x_{4k-8}) A_{k-3}^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}\theta_4} ,$$

and the dihedral ALE spaces are obtained as hyperkähler quotients of M by G/C, where C denotes the circle of scalars in G which acts trivially on M. For generic choice of level set, the action of G/C is free, so the quotient is nonsingular.

We need to modify the above procedure by replacing one of the Euclidean factors by a new hyperkähler space. Now H^4 is a 16-dimensional hyperkähler space with a triholomorphic and isometric $\mathrm{U}(2) \times \mathrm{U}(2)$ action in which the circle of scalars acts trivially. If we replace this by another space with the same properties we shall get new hyperkähler four-manifolds with (presumably) non-ALE metrics. Fortunately a candidate for such a hyperkähler 16-dimensional manifold is at hand.

It has been observed [K3, H3] that if $\mathscr{A}$ is taken to be the infinite-dimensional quaternionic affine space of points $T_0 + \mathrm{i}T_1 + \mathrm{j}T_2 + \mathrm{k}T_3$ where T_i are analytic $\mathrm{u}(2)$-valued functions on $[0, 1]$, then $\mathscr{A}$ is formally hyperkähler. Moreover the group $\hat{G}$ of $\mathrm{U}(2)$-valued functions on $[0, 1]$ acts on $\mathscr{A}$ by

$$T_0 \mapsto g T_0 g^{-1} - \dot{g} g^{-1} , \tag{4}$$

$$T_i \mapsto g T_i g^{-1} \quad (i = 1, 2, 3) , \tag{5}$$

preserving the hyperkähler structure. Here $\dot{g}$ denotes the derivative of g with respect to the coordinate t on $[0, 1]$. Let us define $\tilde{G}$ to be the normal subgroup of $\hat{G}$ consisting of those elements of $\hat{G}$ which are the identity at $t = 0, 1$. The moment map for the action of $\tilde{G}$ is

$$F\colon (T_0, T_1, T_2, T_3) \mapsto \begin{pmatrix} \dot{T}_1 + [T_0, T_1] - [T_2, T_3] \\ \dot{T}_2 + [T_0, T_2] - [T_3, T_1] \\ \dot{T}_3 + [T_0, T_3] - [T_1, T_2] \end{pmatrix} . \tag{6}$$

Using the method of proof of theorem 2.1, adapted to the infinite-dimensional setting, we see that $N = F^{-1}(0, 0, 0)/\tilde{G}$ is hyperkähler. This manifold is just the moduli space of solutions to Nahm's equations [D] for 2×2 analytic matrices.

Nahm's equations are equivalent [D] to

$$\dot{\beta}+[\alpha,\beta]=0\ , \tag{7}$$

$$\dot{\alpha}+\dot{\alpha}^*+[\alpha,\alpha^*]+[\beta,\beta^*]=0\ , \tag{8}$$

where $\alpha=T_0-\mathrm{i}T_1$ and $\beta=T_2+\mathrm{i}T_3$. As pointed out in [K3], the moduli space N can be identified with the quotient by the complexification of $\tilde{G}$ of the space of solutions to eq. (7) alone. This is essentially theorem 2.2, in an infinite-dimensional setting.

Following Kronheimer [K3], we can gauge α to zero by solving the equation $\dot{g}=g\alpha$ subject to the condition $g(1)=\mathrm{Id}$, and the image of β under this gauge transformation is a constant matrix in $\mathfrak{gl}(2,\mathbb{C})$. The map $(\alpha,\beta)\to(\beta(1), g(0))=(B,Q)$ gives a bijection between N and $\mathfrak{gl}(2,\mathbb{C})\times\mathrm{GL}(2,\mathbb{C})$, that is, between N and the cotangent bundle of $\mathrm{GL}(2,\mathbb{C})$.

There is a triholomorphic and isometric action of $\hat{G}/\tilde{G}\cong\mathrm{U}(2)\times\mathrm{U}(2)$ on N. If we identify N with $T^*\mathrm{GL}(2,\mathbb{C})$ as above this action is

$$(B,Q)\mapsto(U_1BU_1^{-1},U_1QU_2^{-1})\ .$$

We see that the action of the scalar circle in $\mathrm{U}(2)\times\mathrm{U}(2)$ is trivial, so we in fact have an $\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2))$ action.

It is clear from our expressions (4), (5) that a point (T_0,T_1,T_2,T_3) of N can only be fixed by the $\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2))$ action if $T_1(0)$, $T_2(0)$, $T_3(0)$ are proportional and $T_1(1)$, $T_2(1)$, $T_3(1)$ are proportional.

There is also an $\mathrm{SO}(3)$ action on N defined by

$$T_0\mapsto T_0\ ,\qquad T_i\mapsto\sum_j a_{ij}T_j\quad(i=1,2,3)\ ,$$

where $(a_{ij})\in\mathrm{SO}(3)$. The action of $\mathrm{SO}(3)$ is isometric, but it permutes the two-sphere of complex structures rather than fixing them.

Finally, there is an isometric $\mathbb{R}^3$ action

$$T_0\mapsto T_0\ ,\qquad T_j\mapsto T_j-\mathrm{i}a_j\,\mathrm{Id}\quad(j=1,2,3)\ ,$$

which fixes complex structures.

We summarise some properties of N as follows.

Proposition 3.1. *N is a hyperkähler manifold of real dimension 16 with a triholomorphic and isometric $\mathrm{U}(2)\times\mathrm{U}(2)$ action in which the scalars act trivially. As a complex manifold, N is isomorphic to $T^*\mathrm{GL}(2,\mathbb{C})$.* □

We can now adapt the construction of [K1] by replacing one H^4 factor by N. This is closely analogous to the way the multi-instanton construction is modified to produce multi-Taub–NUT spaces. There one copy of H is replaced by $\mathbb{R}^3\times\mathrm{S}^1$, which is a moduli space of $\mathfrak{u}(1)$-valued Nahm matrices.

Proposition 3.2. *Let* $M = H^2 \times H^2 \times N \times (H^4)^{k-5} \times H^2 \times H^2$. *Let* $G = \mathrm{U}(1)^4 \times \mathrm{U}(2)^{k-3}$ *act on* M *by*

$$(x_1, x_2) \mapsto (x_1, x_2) A_1^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}\theta_1} ,$$

$$(x_3, x_4) \mapsto (x_3, x_4) A_1^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}\theta_2} ,$$

$$(T_0, T_1, T_2, T_3) \mapsto ((A_1, A_2))_{.} (T_0, T_1, T_2, T_3) ,$$

$$(x_5, x_6, x_7, x_8) \mapsto (x_5, x_6, x_7, x_8)(A_2, A_3) ,$$

$$\dots$$

$$(x_{4k-19}, x_{4k-18}, x_{4k-17}, x_{4k-16}) \mapsto (x_{4k-19}, x_{4k-18}, x_{4k-17}, x_{4k-16})(A_{k-4}, A_{k-3}) ,$$

$$(x_{4k-15}, x_{4k-14}) \mapsto (x_{4k-15}, x_{4k-14}) A_{k-3}^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}\theta_3} ,$$

$$(x_{4k-13}, x_{4k-12}) \mapsto (x_{4k-13}, x_{4k-12}) A_{k-3}^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}\theta_4} .$$

where $((A_1, A_2))_.$ *denotes the* $\mathrm{U}(2) \times \mathrm{U}(2)$ *action on* N.

If C *denotes the subgroup of* G *whose action is trivial then the hyperkähler quotient of* M *by* G/C *is, when non-singular, a four-manifold.*

Proof. C is the circle subgroup of G defined by

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 : A_i = \mathrm{e}^{\mathrm{i}\theta_1} \,\mathrm{Id} \quad (i = 1, \dots, k-3) .$$

The (real) dimension of M is $16k - 32$ and the dimension of G/C is $4k - 9$, so the dimension of the hyperkähler quotient is 4. □

4. The dihedral series

In order to study the hyperkähler quotients more closely we need to calculate the moment maps for the action of G/C on M.

The proof of the following result is very similar to that of theorem 6.1 in [Da1].

Theorem 4.1. *The moment map for the action of* $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$ *on* N *is given by*

$$\mu_i : (T_0, T_1, T_2, T_3) \mapsto \begin{pmatrix} -\langle T_i(0), \sigma_1 \rangle, \langle T_i(1), \sigma_1 \rangle \\ -\langle T_i(0), \sigma_2 \rangle, \langle T_i(1), \sigma_2 \rangle \\ -\langle T_i(0), \sigma_3 \rangle, \langle T_i(1), \sigma_3 \rangle \\ \mathrm{Tr}\, T_i(1) \end{pmatrix} \quad (i = 1, 2, 3) ,$$

where $\sigma_1, \sigma_2, \sigma_3$ *are the Pauli spin matrices, and* $\langle \;,\; \rangle$ *is the standard inner product on* $\mathrm{u}(2)$.

Proof. Every point of N is equivalent under the action of SO(3), $\mathbb{R}^3$ and S(U(2)×U(2)) to one of the form

$$(0, \tfrac{1}{2}f_1\sigma_1, \tfrac{1}{2}f_2\sigma_2, \tfrac{1}{2}f_3\sigma_3) ,$$

where f_1, f_2, f_3 are real-valued functions satisfying the spinning-top equations $\dot{f}_1 = f_2 f_3$ and cyclically. The tangent vectors $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$ to N at this point will satisfy the linearisation of Nahm's equations and the condition of orthogonality to the orbits of the gauge group G. These ordinary differential equations can be solved to give

$$\epsilon_0 = \begin{pmatrix} \dot{f}_1 L_4 \\ \dot{f}_2 L_3 + \tilde{m}_3/f_2 \\ -\dot{f}_3 L_2 - \tilde{n}_2/f_3 \\ \tilde{q}_1 \end{pmatrix}, \qquad \epsilon_1 = \begin{pmatrix} \dot{f}_1 L_1 \\ \dot{f}_2 L_2 + \tilde{m}_2/f_2 \\ \dot{f}_3 L_3 + \tilde{n}_3/f_3 \\ \tilde{q}_2 \end{pmatrix},$$

$$\epsilon_2 = \begin{pmatrix} -\dot{f}_1 L_2 \\ \dot{f}_2 L_1 + \tilde{m}_1/f_2 \\ -\dot{f}_3 L_4 - \tilde{n}_4/f_3 \\ \tilde{q}_3 \end{pmatrix}, \qquad \epsilon_3 = \begin{pmatrix} -\dot{f}_1 L_3 \\ \dot{f}_2 L_4 + \tilde{m}_4/f_2 \\ \dot{f}_3 L_1 + \tilde{n}_1/f_3 \\ \tilde{q}_4 \end{pmatrix},$$

where

$$L_j = \int_0^t \frac{\tilde{m}_j}{f_2^2(s)} + \frac{\tilde{n}_j}{f_3^2(s)}\, \mathrm{d}s + \tilde{p}_j ,$$

and $\tilde{m}_j$, $\tilde{n}_j$, $\tilde{p}_j$, $\tilde{q}_j$ are real numbers ($j = 1, 2, 3, 4$). Here we are identifying u(2) with $\mathbb{R}^4$.

The Kähler forms on the flat infinite-dimensional space $\mathscr{A}$ descend to define Kähler forms ω_i on N. It is now straightforward to calculate the vector fields X for the S(U(2)×U(2)) action on N, evaluate $\omega_i(X, Y)$ for vector fields Y on N, and deduce the required result. Knowing how points of N and tangent vectors to N transform under the actions of $\mathbb{R}^3$, SO(3) and S(U(2)×U(2)) enables us to extend the calculation to arbitrary points of N. □

Under the isomorphism between N and $T^*\mathrm{GL}(2, \mathbb{C})$ we have $B = (T_2 + \mathrm{i}T_3)(1)$ and $Q^{-1}BQ = (T_2 + \mathrm{i}T_3)(0)$, so the complex moment map for the SU(2)×U(2) action is

$$\mu_2 + \mathrm{i}\mu_3 : (B, Q) \mapsto \begin{pmatrix} -\langle Q^{-1}BQ, \sigma_1 \rangle, \langle B, \sigma_1 \rangle \\ -\langle Q^{-1}BQ, \sigma_2 \rangle, \langle B, \sigma_2 \rangle \\ -\langle Q^{-1}BQ, \sigma_3 \rangle, \langle B, \sigma_3 \rangle \\ \mathrm{Tr}\, B \end{pmatrix},$$

or equivalently

$$\mu_2 + i\mu_3 : (B, Q) \mapsto (-Q^{-1}BQ, B) . \tag{9}$$

Let us now consider the hyperkähler quotient $\mu^{-1}(\eta)/(G/C)$, for $\eta \in \mathbb{R}^3 \otimes \mathscr{Z}$ where $\mathscr{Z}$ is the dual of the centre of the Lie algebra of G/C. We defined G to be $U(1)^4 \times U(2)^{k-3}$ so the centre of G is isomorphic to $U(1)^{k+1}$ and the centre of G/C isomorphic to $U(1)^k$. Therefore we can write

$$\eta = (\eta^1, ..., \eta^{k+1}) ,$$

where

$$\eta^s = (\eta_1^s, \eta_2^s, \eta_3^s) \in \mathbb{R}^3 \otimes u(1)^* , \quad s = 1, ..., k+1 ,$$
$$\eta^1 + \cdots + \eta^{k+1} = 0 .$$

We are identifying the Lie algebra of G/C with the trace free elements of the Lie algebra of G.

It follows from our remarks about the dihedral ALE construction, and from our comments above about when the $S(U(2) \times U(2))$ action on N has fixed points, that G/C acts freely on $\mu^{-1}(\eta)$ for generic η. Therefore our hyperkähler quotients are generically nonsingular manifolds.

Let us now specialise to the case when $\eta_1^s = 0$ for all s. Using theorem 2.2, we see that $\mu^{-1}(\eta)/(G/C)$ is then isomorphic as a complex variety to the quotient $(\mu_2 + i\mu_3)^{-1}(\xi)/(G/C)_{\mathbb{C}}$, where

$$\xi = (\xi^1, ..., \xi^{k+1}) ,$$
$$\xi^s = \eta_2^s + i\eta_3^s , \quad s = 1, ..., k+1 .$$

Using our expressions (2), (3) and (9) for the complex moment maps, we can be more explicit; $\mu^{-1}(\eta)/(G/C)$ is isomorphic to the algebro-geometric quotient by $(G/C)_{\mathbb{C}}$ of the variety consisting of elements

$$B \in \mathfrak{gl}(2, \mathbb{C}) , \qquad Q \in GL(2, \mathbb{C}) ,$$
$$R_1, ..., R_4 \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) , \qquad S_1, ..., S_4 \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}) ,$$
$$Z_1, ..., Z_{k-5}, W_1, ..., W_{k-5} \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2) ,$$

satisfying the equations

$$-\mathrm{Tr}\, R_1 S_1 = \xi_1 , \qquad -\mathrm{Tr}\, R_2 S_2 = \xi_2 ,$$
$$B + R_1 S_1 + R_2 S_2 = \xi_3\, \mathrm{Id} , \qquad Z_1 W_1 - Q^{-1} B Q = \xi_4\, \mathrm{Id} ,$$
$$Z_2 W_2 - W_1 Z_1 = \xi_5\, \mathrm{Id} , \qquad Z_3 W_3 - W_2 Z_2 = \xi_6\, \mathrm{Id} ,$$
$$\cdots \qquad \cdots$$
$$Z_{k-5} W_{k-5} - W_{k-6} Z_{k-6} = \xi_{k-2}\, \mathrm{Id} , \qquad R_3 S_3 + R_4 S_4 - W_{k-5} Z_{k-5} = \xi_{k-1}\, \mathrm{Id} ,$$
$$-\mathrm{Tr}\, R_3 S_3 = \xi_k , \qquad -\mathrm{Tr}\, R_4 S_4 = \xi_{k+1} .$$

The action of $(G/C)_{\mathbb{C}}$ here is

$$B \mapsto U_1 B U_1^{-1}, \qquad Q \mapsto U_1 Q U_2^{-1},$$
$$R_1 \mapsto \tau_1^{-1} U_1 R_1, \qquad R_2 \mapsto \tau_2^{-1} U_1 R_2,$$
$$R_3 \mapsto \tau_3^{-1} U_{k-3} R_3, \qquad R_4 \mapsto \tau_4^{-1} U_{k-3} R_4,$$
$$S_1 \mapsto \tau_1 S_1 U_1^{-1}, \qquad S_2 \mapsto \tau_2 S_2 U_1^{-1},$$
$$S_3 \mapsto \tau_3 S_3 U_{k-3}^{-1}, \qquad S_4 \mapsto \tau_4 S_4 U_{k-3}^{-1},$$
$$(Z_i, W_i) \mapsto (U_{i+1} Z_i U_{i+2}^{-1}, U_{i+2} W_i U_{i+1}^{-1}) \quad (i=1, \ldots, k-5),$$

where

$$U_1, \ldots, U_{k-3} \in \mathrm{GL}(2, \mathbb{C}), \qquad \tau_1, \ldots, \tau_4 \in \mathbb{C}^*.$$

We can use the action of $(G/C)_{\mathbb{C}}$ to set Q equal to the identity. Our variety is now the quotient of the space of solutions $(B, R_1, \ldots, R_4, S_1, \ldots, S_4, Z_1, \ldots, Z_{k-5}, W_1, \ldots, W_{k-5})$ to the system of equations

$$-\mathrm{Tr}\, R_1 S_1 = \xi_1, \qquad -\mathrm{Tr}\, R_2 S_2 = \xi_2,$$
$$B + R_1 S_1 + R_2 S_2 = \xi_3 \,\mathrm{Id}, \qquad Z_1 W_1 - B = \xi_4 \,\mathrm{Id},$$
$$Z_2 W_2 - W_1 Z_1 = \xi_5 \,\mathrm{Id}, \qquad Z_3 W_3 - W_2 Z_2 = \xi_6 \,\mathrm{Id},$$
$$\ldots \qquad \ldots$$
$$Z_{k-5} W_{k-5} - W_{k-6} Z_{k-6} = \xi_{k-2} \,\mathrm{Id}, \qquad R_3 S_3 + R_4 S_4 - W_{k-5} Z_{k-5} = \xi_{k-1} \,\mathrm{Id},$$
$$-\mathrm{Tr}\, R_3 S_3 = \xi_k, \qquad -\mathrm{Tr}\, R_4 S_4 = \xi_{k+1},$$

by the action

$$B \mapsto U_1 B U_1^{-1},$$
$$R_1 \mapsto \tau_1^{-1} U_1 R_1, \qquad R_2 \mapsto \tau_2^{-1} U_1 R_2,$$
$$R_3 \mapsto \tau_3^{-1} U_{k-3} R_3, \qquad R_4 \mapsto \tau_4^{-1} U_{k-3} R_4,$$
$$S_1 \mapsto \tau_1 S_1 U_1^{-1}, \qquad S_2 \mapsto \tau_2 S_2 U_1^{-1},$$
$$S_3 \mapsto \tau_3 S_3 U_{k-3}^{-1}, \qquad S_4 \mapsto \tau_4 S_4 U_{k-3}^{-1},$$
$$(Z_i, W_i) \mapsto (U_{i+1} Z_i U_{i+2}^{-1}, U_{i+2} W_i U_{i+1}^{-1}) \quad (i=1, \ldots, k-5),$$

where we now have $U_1 = U_2$.

The map $(B, R_i, S_i, Z_i, W_i) \mapsto (R_i, S_i, Z_i, W_i)$ induces an isomorphism onto the quotient of the variety of solutions $(R_1, \ldots, R_4, S_1, \ldots, S_4, Z_1, \ldots, Z_{k-5}, W_1, \ldots, W_{k-5})$ to the equations

$$-\mathrm{Tr}\, R_1 S_1 = \xi_1 \,, \qquad -\mathrm{Tr}\, R_2 S_2 = \xi_2 \,,$$

$$Z_1 W_1 + R_1 S_1 + R_2 S_2 = (\xi_3 + \xi_4)\ \mathrm{Id} \,,$$

$$Z_2 W_2 - W_1 Z_1 = \xi_5\ \mathrm{Id} \,, \qquad Z_3 W_3 - W_2 Z_2 = \xi_6\ \mathrm{Id} \,,$$

$$\ldots \qquad \ldots$$

$$Z_{k-5} W_{k-5} - W_{k-6} Z_{k-6} = \xi_{k-2}\ \mathrm{Id} \,, \qquad R_3 S_3 + R_4 S_4 - W_{k-5} Z_{k-5} = \xi_{k-1}\ \mathrm{Id} \,,$$

$$-\mathrm{Tr}\, R_3 S_3 = \xi_k \,, \qquad -\mathrm{Tr}\, R_4 S_4 = \xi_{k+1} \,,$$

by the action

$$R_1 \mapsto \tau_1^{-1} U_2 R_1 \,, \qquad R_2 \mapsto \tau_2^{-1} U_2 R_2 \,,$$

$$R_3 \mapsto \tau_3^{-1} U_{k-3} R_3 \,, \qquad R_4 \mapsto \tau_4^{-1} U_{k-3} R_4 \,,$$

$$S_1 \mapsto \tau_1 S_1 U_2^{-1} \,, \qquad S_2 \mapsto \tau_2 S_2 U_2^{-1} \,,$$

$$S_3 \mapsto \tau_3 S_3 U_{k-3}^{-1} \,, \qquad S_4 \mapsto \tau_4 S_4 U_{k-3}^{-1} \,,$$

$$(Z_i, W_i) \mapsto (U_{i+1} Z_i U_{i+2}^{-1}, U_{i+2} W_i U_{i+1}^{-1}) \quad (i = 1, \ldots, k-5) \,.$$

However, this is just the complex manifold which we obtain from the D_{k-1} construction of ALE spaces in [K1], so we have established the following result.

Theorem 4.2. *The hyperkähler quotients of M by G/C are deformations of the D_{k-1} Kleinian dihedral singularity*

$$x^2 - zy^2 = z^{k-2}$$

for $k \geqslant 5$. □

The above construction produces hyperkähler deformations of the D_k singularities for $k \geqslant 4$. We can also treat the D_2 and D_3 cases as follows.

For D_3, we take $M = N \times H^2 \times H^2$ and $G = \mathrm{U}(1) \times \mathrm{U}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$. The action is

$$(T_0, T_1, T_2, T_3) \mapsto ((A_1, A_2)).(T_0, T_1, T_2, T_3) \,,$$

$$(x_1, x_2) \mapsto (x_1, x_2) A_2^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}\theta_1} \,,$$

$$(x_3, x_4) \mapsto (x_3, x_4) A_2^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}\theta_2} \,,$$

where A_1 lies in a non-central U(1) subgroup of U(2), and A_2 lies in U(2).

Similar arguments to those above show that the resulting hyperkähler quotients are biholomorphic to hyperkähler quotients of H^4 by $\mathrm{U}(1)^3$ where the action is that of (1). These complex manifolds are deformations of the A_3 singularity $xy = z^4$, which is equivalent to the D_3 singularity $x^2 - zy^2 = z^2$ (recall that the Dynkin diagrams D_3 and A_3 are identical).

For D_2 we proceed as follows. The hyperkähler quotient of N by the centre of $S(U(2)\times U(2))$ is a hyperkähler 12-manifold N^0 with a triholomorphic and isometric $SU(2)\times SU(2)$ action. As a complex manifold, N^0 can be identified with $T^*SL(2,\mathbb{C})$.

Let us now consider the diagonal $U(1)\times U(1)$ subgroup of $SU(2)\times SU(2)$ associated to the $U(1)$ subgroup of $SU(2)$ which stabilises the Pauli spin matrix $\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ in the adjoint representation. We shall take the hyperkähler quotient of N^0 by this torus action.

Of course, we could perform a similar construction for any other $U(1)\times U(1)$ subgroup of $SU(2)\times SU(2)$. However, the resulting hyperkähler four-manifolds are in fact isometric via the $SU(2)\times SU(2)$ action on N^0 to those obtained with the choice of $U(1)\times U(1)$ made above.

From theorem 4.1 the moment map for the action of $U(1)\times U(1)$ is given by

$$\mu\colon (T_0, T_1, T_2, T_3) \mapsto \begin{pmatrix} (-\langle T_1(0), \sigma_1\rangle, \langle T_1(1), \sigma_1\rangle) \\ (-\langle T_2(0), \sigma_1\rangle, \langle T_2(1), \sigma_1\rangle) \\ (-\langle T_3(0), \sigma_1\rangle, \langle T_3(1), \sigma_1\rangle) \end{pmatrix},$$

and the associated complex moment map is

$$(B, Q) \mapsto (-\langle Q^{-1}BQ, \sigma_1\rangle, \langle B, \sigma_1\rangle) . \tag{10}$$

Consider the hyperkähler quotient

$$M(u, v) = \mu^{-1}((0, 2\,\mathrm{Re}\, u, 2\,\mathrm{Im}\, u), (0, 2\,\mathrm{Re}\, v, 2\,\mathrm{Im}\, v))/U(1)\times U(1) .$$

The factor of 2 is for convenience. It easily follows from theorem 2.2 that $M(u, v)$ is isomorphic as a complex manifold to the quotient by $\mathbb{C}^*\times\mathbb{C}^*$ of

$$\{(B, Q)\in \mathfrak{sl}(2,\mathbb{C})\times SL(2,\mathbb{C}) : \mathrm{str}(B) = -2iv,\ \mathrm{str}(Q^{-1}BQ) = 2iu\} , \tag{11}$$

where

$$\mathrm{str}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} - a_{22} .$$

The $\mathbb{C}^*\times\mathbb{C}^*$ action is given by

$$B \mapsto \phi B \phi^{-1} , \qquad Q \mapsto \phi Q \psi^{-1} ,$$

where $\phi = \mathrm{diag}(\tau, \tau^{-1})$, $\psi = \mathrm{diag}(\rho, \rho^{-1})$ and $\tau, \rho \in \mathbb{C}^*$. It is easily checked that fixed points occur only if B is diagonal and either

$$Q = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \quad \text{with} \quad \phi = \psi$$

or

$$Q = \begin{pmatrix} 0 & q \\ -q^{-1} & 0 \end{pmatrix} \quad \text{with} \quad \phi = -\psi .$$

In these cases, we find that $\operatorname{str}(B) = \pm\operatorname{str}(Q^{-1}BQ)$, respectively, so $u = \pm v$. Let us assume from now on that $u \neq \pm v$. With this restriction, we can show that all points of the variety (11) are stable with respect to the action of $\mathbb{C}^* \times \mathbb{C}^*$.

We shall now identify the gravitational instantons $M(u, v)$ with hypersurfaces in $\mathbb{C}^3$. Let us put

$$Q = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad B = \begin{pmatrix} -\mathrm{i}v & a \\ b & \mathrm{i}v \end{pmatrix}.$$

The map

$$(B, Q) \mapsto (X_0, X_1, X_2, X_3, X_4) = (a, b, ps, pq, rs)$$

defines an isomorphism between $M(u, v)$ and the quotient of the space

$$\{(X_0, X_1, X_2, X_3, X_4) : X_2^2 - X_2 = X_3 X_4 ,$$
$$(1 - 2X_2)\mathrm{i}v - \mathrm{i}u = X_1 X_3 - X_0 X_4\} \qquad (12)$$

by the $\mathbb{C}^*$ action

$$(X_0, X_1, X_2, X_3, X_4) \mapsto (\tau^2 X_0, \tau^{-2} X_1, X_2, \tau^2 X_3, \tau^{-2} X_4) .$$

Now consider the map

$$(X_0, X_1, X_2, X_3, X_4) \mapsto (Y_0, Y_1, Y_2) = (X_2, X_0 X_4, X_0 X_1) .$$

This map gives an isomorphism of $M(u, v)$ with the variety

$$\{(Y_0, Y_1, Y_2) : Y_2(Y_0^2 - Y_0) = ((1 - 2Y_0)\mathrm{i}v - \mathrm{i}u + Y_1) Y_1\} .$$

Putting $x = Y_1$, $y = \mathrm{i}(2Y_0 - 1)$, $z = -\frac{1}{4}Y_2$ we get the hypersurface in $\mathbb{C}^3$ with equation

$$x^2 - zy^2 = z + vxy + \mathrm{i}ux . \qquad (13)$$

This is a nonsingular affine surface (if $u \neq \pm v$) and is a deformation of the D_2 singularity

$$x^2 - zy^2 = z . \qquad (14)$$

The surface with eq. (14) has *two* singularities, at $(0, \pm\mathrm{i}, 0)$, and is nonsingular at the origin.

Hitchin [H2] argued using twistor methods that the D_2 singularity should admit deformations carrying nonsingular hyperkähler structures (this was also noted on heuristic grounds by Page [P]).

5. Remarks

Our gravitational instantons are deformations of the singular hypersurface in $\mathbb{C}^3$ whose equation is

$$x^2 - zy^2 = z^{k-1} . \tag{15}$$

If $k=1$, eq. (15) defines a *nonsingular* complex surface. This, in fact, is the double cover of the Atiyah–Hitchin manifold [AH] and hence also admits a hyperkähler structure. A one-parameter family of hyperkähler deformations of this space is known [Da2].

The hyperkähler deformations of the D_k singularity are obtained as hyperkähler quotients by a group with a $(k+1)$-dimensional centre, so there are $3k+3$ parameters for choice of level set. However, not all of these choices give geometrically distinct metrics. The SO(3) action on N, combined with left multiplication of H^2 and H^4 by unit quaternions, induces an action of SO(3) on M. Also there is an $\mathbb{R}^3$ action on M induced by the action $T_j \mapsto T_j - \mathrm{i}a_j$ Id on N. These two actions on M map level sets of our moment map isometrically onto other level sets, so the number of effective parameters for the family of hyperkähler quotients is $3k-3$ if $k \geqslant 2$.

Generically, the gravitational instantons we have constructed will have no isometries. However, if the vectors η^i in the choice of level set are collinear in $\mathbb{R}^3$, there will be an isometric nontriholomorphic action of U(1) inherited from the actions of SO(3) and $\mathbb{R}^3$ on M.

What is not clear from the construction is the asymptotic behaviour of our manifolds. As N is a moduli space of solutions to Nahm's equations we expect it to be isometric to a moduli space of monopoles on $\mathbb{R}^3$ and hence have some sort of asymptotic flatness property, which might be shared by our gravitational instantons. However, it seems difficult to prove this from our hyperkähler quotient picture.

Finally, it is natural to ask if a similar modification of the construction of ALE gravitational instantons is possible in the case of the exceptional singularities E_6, E_7, E_8. The Euclidean factors occurring in the quotient construction here are of the form H^{mn} with triholomorphic and isometric actions of $\mathrm{U}(m) \times \mathrm{U}(n)$ in which the circle of scalars acts trivially. The pairs (m, n) which occur are

(i) (1, 2), (2, 3) for E_6,

(ii) (1, 2), (2, 3), (3, 4), (2, 4) for E_7,

(iii) (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (4, 6), (2, 4), (3, 6) for E_8.

Unfortunately the author is not aware of a non-Euclidean hyperkähler space of real dimension $4nm$ with a triholomorphic and isometric action of $\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$ for any of these values of (m, n), so is unable to produce new examples of hyperkähler four-manifolds associated to the exceptional groups.

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